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Interaction of waves with two-dimensional obstacles: a relation between the radiation and scattering problems

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A relation connecting the reflexion and transmission coefficients for scattering of water waves by a fixed body with the far-field radiated waves due to forced motions of the same body is derived. Two alternative derivations are given, including a simple argument based on the analysis of an appropriate linear superposition of the two problems, and a more formal application of Green's theorem to the two potentials. For bodies with horizontal symmetry, the transmission and reflexion coefficients are related to the phase angles of the far-field radiated waves associated with symmetric and antisymmetric forced motions of the body. Some general conclusions follow for arbitrary symmetric bodies, and these are verified in specific cases by comparison with existing solutions. The applicability of these relations to other types of wave problem is noted.

1. Introduction

It is customary to call the interaction of incident plane waves with a fixed body a 'scattering' problem and the generation of waves by forced oscillatory motion of the body in an otherwise undisturbed fluid a 'radiation' problem. From the mathematical viewpoint these two problems differ in terms of the boundary condition on the body and the far-field conditions. From the physical viewpoint scattering and radiation appear to be unrelated, except in long-wavelength approximations, where the local influence of the body on the surrounding fluid is the same for both problems.

In spite of the apparent differences between the scattering and radiation problems, there exist certain relationships between them which are consequences of reciprocity principles and Green's theorem. A particularly useful example, known as 'Haskind's relations' in ship hydrodynamics, provides a linear relation between the exciting forces exerted by incident waves on a fixed body and the amplitude of the far-field radiated waves generated by forced motions of the body in otherwise calm water. From considerations of the energy flux at infinity, one can then obtain a relation between the damping force in calm water and the exciting force in incident waves, as shown by Newman (1962). A general review of this subject has been given by Ogilvie (1973).

In this paper attention is focused on the reflexion and transmission coefficients of the scattering problem, for two-dimensional motion involving the interaction of an incident plane progressive wave system with a long cylindrical body. The body generators are horizontal and parallel (or possibly oblique) to the wave crests. In this situation a portion of the incident wave energy is reflected by the body as an 'upstream' propagating wave, while the remainder is transmitted past the body as a 'downstream' propagating wave, the reflexion and transmission coefficients being defined as the ratios of the amplitudes of these two wave systems to the amplitude of the incident wave. In order to relate these ratios to properties of the radiation problem, we shall first construct in § 3 a linear combination of the scattering and radiation solutions such that there is no net wave system downstream of the body, linearization being assumed throughout so that superposition of solutions is valid. The resulting 'composite' solution satisfies a boundary-value problem with the same boundary condition on the body as for the radiation problem (since the scattering solution involves a homogeneous boundary condition on the body), zero wave propagation at downstream infinity (by definition of the composite solution) and a prescribed combination of incident and reflected waves at upstream infinity.

On restricting the normal velocity on the body to be of fixed phase, it follows that the part of the composite solution with conjugate phase satisfies a boundary-value problem which is homogeneous except for the presence of upstream standing waves. It is then argued heuristically that, unless the transmission coefficient of the body is zero, the upstream waves of this otherwise homogeneous problem must vanish. This provides a simple relation between the far-field characteristics of the radiation and scattering problems. The same relation is confirmed in § 4 by a more formal derivation based on Green's theorem.

We seek then to determine the reflexion and transmission coefficients from the solution of the radiation problem. In fact, the radiation problem is non-unique without specification of the particular forced motion of the body, and it is necessary to consider two independent radiation problems. Attention is restricted in § 5 to bodies which are symmetric about the vertical (y) axis, and the appropriate independent radiation problems are respectively symmetric and antisymmetric in x, i.e. those for forced vertical and horizontal motions of the body. The reflexion and transmission coefficients can then be expressed in terms of the phase angles δ_s and δ_a of the symmetric and antisymmetric radiated waves.

Various conclusions follow, both for general and specific body geometries. Since the reflexion and transmission coefficients are unique properties of the body geometry and wavelength, the same must be true of the phase angles δ_s and δ_a, without specification of the forced motions of the body. Thus every symmetric (antisymmetric) forced motion of the body will generate radiated waves with the same phase angle $\delta_s (\delta_a)$, subject to a possible shift of $\pm 180°$. Since, from the Haskind relations, the corresponding forces or moments are directly proportional to the complex amplitudes of the radiated waves, similar conclusions apply to the phase angles of the exciting forces and moment. In particular, the horizontal exciting force is in phase with the exciting moment about the roll axis. For specific body shapes it is possible to verify these results by comparison with existing calculations of the reflexion and transmission coefficients, the exciting forces and moment, and the characteristics of the radiated waves. Such comparisons are made in § 6 for a submerged circular cylinder and in § 7 for floating or submerged vertical barriers.

Our derivation is based on consideration of two-dimensional surface waves, but since the analysis depends only on a simple superposition of the far-field waves and is independent of the governing partial differential equation, similar conclusions hold for other types of linear wave problems, including water waves at oblique incidence, acoustic waves in a waveguide below the cut-off frequency and two-dimensional internal waves at frequencies below the Brunt–Väisälä frequency. In the latter context it may be noted that our analysis is similar in certain respects to that of Drazin & Moore (1967), who used an analogous superposition of two wave systems in order to satisfy the radiation conditions for steady flow past an obstacle, and that the same scheme has been exploited in a numerical solution of the steady-state ship-wave problem by Mei & Chen (1974).

2. The radiation and scattering problems

Assume that a long cylindrical body with horizontal generators is situated on or beneath the free surface of a fluid of constant or infinite depth. For the sake of definiteness we assume that the fluid motion is two-dimensional and confined to planes normal to the cylinder generators, but the results which follow are identical in the more general instance where the motion is sinusoidal in the direction parallel to the cylinder, as in the case of a cylinder with oblique incident waves. The problem is assumed to be linearized, with oscillatory time dependence of frequency $\omega/2\pi$. Thus the velocity potential can be written in the form

$$\Phi(x,y,t) = \mathrm{Re}\,[\phi(x,y)\,e^{i\omega t}], \tag{1}$$

with a similar representation for the free-surface elevation. Here (x, y) are Cartesian co-ordinates, taken in the usual sense with x horizontal and positive to the right. The potential $\phi(x,y) \equiv \chi(x,y) + i\psi(x,y)$ is complex valued to represent the magnitude and phase of the oscillatory motion.

The velocity potential is governed by Laplace's equation throughout the interior of the fluid domain, with appropriate boundary conditions on the free surface and lower boundary of the fluid. However, these aspects of the boundary-value problem do not concern us explicitly here, and we need consider only the remaining boundary conditions, those on the body and at infinity.

In the *radiation* problem a normal velocity $\mathrm{Re}\,(fe^{i\omega t})$ is prescribed on the body surface, with f a given function of position on the body. Denoting the radiation potential by ϕ_r, the appropriate boundary condition on the body is

$$\partial\phi_r/\partial n = f. \tag{2}$$

Suitable radiation conditions must be imposed at large distances from the body, namely that the waves are outgoing, or that on the free surface

$$\phi_r \sim A_\pm e^{\mp iKx}, \quad x \to \pm\infty. \tag{3}$$

Here $A_{\pm}$ are complex coefficients, representing the wave amplitude and phase at infinity.

In the *scattering* problem, with potential ϕ_s, the body is stationary and waves are incident from infinity. Hence the boundary conditions are that, on the body,

$$\partial\phi_s/\partial n = 0, \tag{4}$$

and at infinity on the free surface,

$$\phi_s \sim \begin{cases} e^{iKx} + R\,e^{-iKx}, & x \to +\infty, \quad (5a) \\ T\,e^{iKx}, & x \to -\infty. \quad (5b) \end{cases}$$

Here e^{iKx} denotes the incident wave, assumed to be of unit amplitude and propagating from $x = +\infty$, whereas R and T are the (complex) reflexion and transmission coefficients as defined in § 1.

3. The composite problem

Let us assume that $T \neq 0$ and define the *composite* potential

$$\phi_c = \phi_r - (A_-/T)\,\phi_s \tag{6}$$

such that there are no waves at $x = -\infty$. This potential satisfies the boundary-value problem stated in § 2 above except that on the body, from (2) and (4),

$$\partial\phi_c/\partial n = f, \tag{7}$$

whereas at infinity on the free surface, from (3) and (5),

$$\phi_c \sim \begin{cases} (A_+ - A_-R/T)\,e^{-iKx} - (A_-/T)\,e^{iKx}, & x \to +\infty, \quad (8a) \\ 0, & x \to -\infty. \quad (8b) \end{cases}$$

The forced motion of the body surface is arbitrary, but we shall restrict f to be real, so that the normal velocity of the body is in phase with $\pm \cos\omega t$. It then follows that the imaginary part ψ_c of the composite potential ϕ_c satisfies a homogeneous boundary condition on the body, as well as at downstream infinity, and the only inhomogeneity in the boundary-value problem for ψ_c is that at upstream infinity it must be equal to a standing wave given by the imaginary part of (8a). Thus the potential ψ_c corresponds to the problem where a standing wave is present at $x = +\infty$ *and the body is stationary*. Equivalently, ψ_c is the solution of the scattering problem with zero transmission coefficient, $T = 0$, and complete reflexion, $|R| = 1$, in contradiction to our assumption that $T \neq 0$. This leads us to one of two possible conclusions: either the transmission coefficient is zero or the potential ψ_c must vanish at *both* infinities.

The occurrence of complete reflexion and zero transmission is generally presumed to be impossible except for pathological body geometries and special wavelengths of the incident wave. The only known examples of complete reflexion (excluding the trivial case where the body completely blocks off the flow) involve situations with a resonant cavity of some sort; hence complete reflexion is known to occur for water-wave scattering by a pair of parallel obstacles at discrete eigenfrequencies where the fluid region between the two obstacles is resonant. This phenomenon is discussed by Evans & Morris (1972*a*), Evans (1974) and Newman (1974).

Below we shall ignore special cases of this nature, and assume that the body geometry and wavelength are such that complete reflexion is impossible. This can be reconciled with the boundary-value problem for $\psi_c = \mathrm{Im}(\phi_c)$ if and only if the upstream standing wave of that problem vanishes, and hence the imaginary part of (8*a*) must be zero.† Thus it follows that

$$\mathrm{Im}[(A_+ - A_- R/T)\, e^{-iKx} - (A_-/T)\, e^{iKx}] = 0, \tag{9}$$

or for this relation to be independent of x,

$$A_+ - A_- R/T + A_-^*/T^* = 0, \tag{10}$$

where an asterisk denotes a complex conjugate. Equation (10) is the principal result of our analysis, and provides a relation between the far-field radiated wave amplitudes $A_\pm$ and the reflexion and transmission coefficients R and T.

An alternative relation to (10) can be obtained by multiplying its conjugate by R, and adding this product to (10). It follows that

$$A_+ + A_+^* R + A_-^*(1 - RR^*)/T^* = 0. \tag{11}$$

From conservation of energy in the scattering problem,

$$RR^* + TT^* = 1. \tag{12}$$

Using (12) to replace the factor in parentheses in (11) gives the desired expression:

$$A_+ + A_+^* R + A_-^* T = 0. \tag{13}$$

4. Derivation based on Green's theorem

While the derivation in § 3 is very simple, its heuristic nature is obvious and one may prefer a more formal mathematical approach. For this purpose we define the operator

$$I(\phi, \psi) = \int_C (\phi\, \partial\psi/\partial n - \psi\, \partial\phi/\partial n)\, dl, \tag{14}$$

where C is the closed contour including the free surface, body surface, fluid bottom and two vertical closures at $x = \pm\infty$. By Green's theorem $I \equiv 0$ for any pair of functions ϕ and ψ which are harmonic in the fluid region bounded by C, and if these functions satisfy the appropriate boundary conditions on the free surface and bottom there is no contribution from those portions of C. By straightforward reduction using the radiation and boundary conditions (2)–(5), it follows that

$$I(\phi_s, \phi_s^*) = 0 = -2iMK(1 - RR^* - TT^*), \tag{15}$$

$$I(\phi_r, \phi_s) = 0 = 2iMKA_+ - \int_{C_B} f\phi_s\, dl, \tag{16}$$

$$I(\phi_r^*, \phi_s) = 0 = -2iMK(\mathrm{A}_+^* R + A_-^* T) - \int_{C_B} f^* \phi_s\, dl. \tag{17}$$

† In general ψ_c must vanish throughout the fluid, but this additional conclusion is not required here, and depends on a uniqueness proof which has only been established for special body geometries.

Here C_B denotes the body contour and M is a positive real constant, defined for water waves as

$$M = (\operatorname{sech}^2 Kh)\int_{-h}^{0} \cosh^2 Ky\,dy = \frac{2Kh+\sinh 2Kh}{4K\cosh^2 Kh}. \tag{18}$$

Equation (15) gives the energy relation (12), expressing conservation of energy flux in the scattering problem. Equation (16) is the general two-dimensional form of Haskind's relations and, by appropriate choice of the function f, can be used to find the scattered wave force

$$\mathbf{F}_s = \int_{C_B} p_s \mathbf{n}\,dl = -i\omega\rho\int_{C_B} \phi_s \mathbf{n}\,dl \tag{19}$$

and a corresponding expression for the moment in terms of the radiated wave amplitude A_+. (Here ρ is the fluid density, p_s the pressure in the scattering problem, and the linearized Bernoulli equation has been used.)

Subtracting (17) from (16) gives the relation

$$I(\phi_r-\phi_r^*, \phi_s) = 0 = 2iMK(A_+ + A_\mp^* R + A_\pm^* T) - \int_{C_B} (f-f^*)\,\phi_s\,dl. \tag{20}$$

Now if f is restricted to be real, as in § 3, the integral in (20) vanishes and (13) is obtained. This derivation of (13) remains valid if $T = 0$, in contrast to the analysis of § 3.

5. Symmetric obstacles

In order to simplify the application of (13) we shall restrict our attention to bodies which are symmetric about the plane $x = 0$. It follows that the two radiation problems in which the normal velocity distribution on the body, represented by the function f, is respectively symmetric and antisymmetric may be considered separately. In the symmetric case, exemplified by vertical oscillations of the body, the radiation potential ϕ_r is an even function of x, and thus

$$A_+ = A_- \equiv A_s. \tag{21}$$

In the antisymmetric case, exemplified by horizontal oscillations of the body, the radiation potential is an odd function of x, and thus

$$A_+ = -A_- \equiv A_a. \tag{22}$$

Applying (13) separately in each case gives the equations

$$R+T = -A_s/A_s^* \equiv -\exp(2i\delta_s), \tag{23}$$

$$R-T = -A_a/A_a^* \equiv -\exp(2i\delta_a). \tag{24}$$

Hence the reflexion and transmission coefficients can be expressed in terms of the phase angles $\delta_{s,a} = \arg A_{s,a}$ of the symmetric and antisymmetric radiated waves in the form

$$R = -\tfrac{1}{2}[\exp(2i\delta_a)+\exp(2i\delta_s)], \tag{25}$$

$$T = \tfrac{1}{2}[\exp(2i\delta_a)-\exp(2i\delta_s)]. \tag{26}$$

Equations (25) and (26) can be replaced by parametric relations

$$R = -\cos\alpha\, e^{i\beta}, \quad T = i\sin\alpha\, e^{i\beta}, \tag{27), (28}$$

where

$$\alpha = \delta_a - \delta_s, \quad \beta = \delta_a + \delta_s. \tag{29), (30}$$

In this form the energy relation (12) is obvious, as is the phase-angle relation $|\arg R - \arg T| = \frac{1}{2}\pi$, which was derived by Newman (1965, equation (2.20)). From (29) and (30) it is apparent that the parameters α and β, which determine the magnitude and phase, respectively, of the reflexion and transmission coefficients, depend uniquely on the phase angles δ_s and δ_a of the radiation problem, and vice versa. Thus a knowledge of both the symmetric *and* antisymmetric radiation phase angle is necessary, in general, to determine R or T. This limits the utility of our relations to those problems where solutions of both the symmetric and antisymmetric radiation problems have been found.

On the other hand, two general conclusions follow which add significantly to the value of these relations. First note that we have dealt rather abstractly with a symmetric and antisymmetric radiation potential, specifying only that the normal velocity on the body surface should be respectively even or odd in x, and with the added restriction that this normal velocity must be of constant phase proportional to $\pm\cos\omega t$ (i.e. f must be real). There are an infinite number of modes of body motions possible in each case, and yet it is clear from (27)–(30) that, since R and T are uniquely specified by the body shape and frequency, the same must be true of the radiation phase angles δ_s and δ_a, with a possible ambiguity of $\pm 180°$. Thus it follows that the phase of the radiated waves takes the same value, δ_s, for all possible symmetric modes of body motion and the same value, δ_a, for all possible antisymmetric modes, modulo 180°. In particular, vertical oscillations of the body and symmetric source-like dilations of the body will produce radiated waves of the same phase, δ_s; likewise, horizontal oscillations and rolling oscillations will produce radiated waves of the same phase angle, $\pm\delta_a$, at $x = \pm\infty$. Moreover, we recall that the Haskind relations (16) and (19) relate the amplitude *and phase* of the exciting force or moment exerted on the body in the scattering problem to the amplitude *and phase* of the radiated waves for forced motions of the body in otherwise calm water, the mode of the forced motions corresponding to the appropriate component of the exciting force or moment. In fact, the complex exciting force or moment is directly proportional to the complex amplitude of the radiated wave, with a constant of proportionality that is real if the phase is related to the incident wave amplitude at the origin. Thus the phase angles δ_s and δ_a are identical to the phase angles of the exciting forces and moments. In particular, δ_s is the phase of the vertical exciting force, and both the horizontal force and rolling moment must have the same phase angle, δ_a.

6. The submerged circular cylinder

A particular scattering problem of interest is the reflexion and transmission of surface waves by a submerged circular cylinder in water of infinite depth. Dean (1948) showed that for this case the reflexion coefficient $R = 0$, the waves being totally transmitted but with a phase shift δ_T. Ursell (1950) re-examined the problem, setting it on a more rigorous basis and outlining a practical scheme for computing the velocity potential. Ogilvie (1963) has presented a complete numerical solution, including the magnitudes and phases of the vertical and horizontal forces acting upon the cylinder in both the scattering and radiation problems.

Since $R = 0$, (27)–(30) imply the relations

$$\delta_a - \delta_s = \tfrac{1}{2}\pi, \quad \delta_T = 2\delta_a. \tag{31), (32}$$

From (31) the phase of the antisymmetric radiated waves, and hence also the horizontal exciting force, will lead the symmetric radiated waves and vertical exciting force by a quarter-period. This is consistent with Ogilvie's (1963) equation (27). From (32) (and the Haskind relations), the phase of the transmitted wave is twice the phase of the horizontal exciting force, and this is confirmed by Ogilvie's (1963, p. 467) analysis, which includes a physical explanation concerning the plausibility of this result. Thus specific results of Ogilvie (1963) confirm our relations (27)–(30), but the completeness of his treatment of the problem precludes additional conclusions or extensions here.

7. The vertical flat plate

A number of analytic solutions have been carried out for scattering and radiation from a vertical flat plate which is either intersecting the free surface or completely submerged. In this case the reflexion coefficient is non-zero, offering a better opportunity to test our relationships.

For this problem the vertical wave force is zero (in the linear theory), and there is no wave radiation associated with vertical oscillations. Nevertheless, the symmetric phase angle δ_s can be inferred by recalling that any symmetric mode of forced motion will suffice to determine δ_s. A non-trivial mode is that of symmetric expansion of the vertical plate, and in the simplest example (corresponding to a delta-function mode of expansion) we may replace the plate by a point source of oscillating strength. From the known expressions for an oscillating source beneath a free surface (cf. Wehausen & Laitone 1960, equation 13.31 or 13.34 for infinite or finite depth, respectively), it can be verified that the phase angle $\delta_s = \frac{1}{2}\pi$. In fact, the same result follows from a more elementary calculation based upon the plane-wave potential, for if $f(y)$ is a specified horizontal velocity on the vertical axis, it follows from our argument regarding the uniqueness of phase angles that δ_s is independent of the choice of $f(y)$ so long as this function is real. (This fact is confirmed by the 'wave-maker theory' of Havelock 1929.) The simplest choice for $f(y)$ is the horizontal velocity of a plane wave. The appropriate outward-radiating wave for positive x is proportional to e^{-iKx} and, after differentiation with respect to x, we conclude that the phase angle relating the radiated wave potential to the normal velocity on the flat plate is $\frac{1}{2}\pi$.

With $\delta_s = \frac{1}{2}\pi$, it follows immediately from (23) that

$$R + T = 1. \tag{33}$$

This property is well known for the surface-piercing vertical plate (cf. Wehausen

& Laitone 1960, p. 532) and can be confirmed for the submerged case from Evans' (1970) equations (38) and (39). Our analysis shows that (33) is also valid for the case of finite depth and for the scattering problem of oblique waves studied by Evans & Morris (1972*b*). The phase angles of the horizontal exciting force and rolling moment, $\delta_a = \alpha + \frac{1}{2}\pi$, also can be compared in the above problems: our relations are consistent with Evans' (1970) equations (73) and (74) for the submerged case and with corresponding results obtained by Kotik (1963) for the surface-piercing flat plate. In the light of our conclusions regarding the uniqueness of δ_a, it is more readily understood why the forced rolling and horizontal oscillation problems for vertical obstacles have the same phase angle and why, for example, there exists for such bodies a vertically displaced point of rotation such that rolling oscillations about this point do not radiate waves at infinity. In fact, we observe that these properties apply to more general bodies and for finite depths as well.

8. Summary and conclusions

An expression relating the reflexion and transmission coefficients of the scattering problem to the phases of symmetric and antisymmetric radiated waves has been derived. In conjunction with the Haskind relations it is possible also to relate the reflexion and transmission coefficients to the exciting forces acting on the body in the same scattering problem. Alternatively, both the reflexion and transmission coefficients, and the exciting forces, can be derived from the properties of the radiation problem, making it unnecessary to solve the scattering problem for these coefficients and forces. The results are limited to two-dimensional linear wave propagation, but apart from this restriction they are quite general and can be applied to other problems such as the scattering of water waves with oblique incidence, where the governing equation is a modified two-dimensional wave equation, and acoustic or internal waves in the regimes where only one radiating wave component is present.

Using these relations one can verify or extend the calculations of several specific water-wave problems. In §§6 and 7 this was done for a submerged circular cylinder and a vertical flat plate. Other body shapes have been treated numerically, and it is possible to apply our results to these too, but the list of references is already lengthy, and the general ideas which follow from our relations should be sufficiently clear from the examples already chosen.

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